

## Two-loop QED corrections to Bhabha scattering

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ABSTRACT: We obtain a simple relation between massless and massive scattering amplitudes in gauge theories in the limit where all kinematic invariants are large compared to particle masses. We use this relation to derive the two-loop QED corrections to large-angle Bhabha scattering.

KEYWORDS: NLO Computations, Electromagnetic Processes and Properties.

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## 1. Introduction

Bhabha scattering,  $e^+e^- \rightarrow e^+e^-$ , is one of the basic processes at electron-positron colliders. It has sufficiently large cross-section to be employed as a reference process for collider luminosity measurements. To determine the luminosity, one takes the ratio of the number of electron-positron pairs observed in the detector and the cross-section for Bhabha scattering computed theoretically. It follows that the accuracy of the luminosity determination is fully correlated with the precision of the theoretical description of Bhabha scattering.

There are two kinematic regimes relevant for Bhabha scattering. *Small angle* scattering,  $\theta \lesssim$  few degrees, was employed at LEP and SLC for the luminosity determination, while *large angle* Bhabha scattering,  $\theta \gtrsim 10$  degrees, is used at flavor factories (BABAR, BELLE, BEPC-BES, CLEO-C, DAPHNE, VEPP-2M) for that purpose [1]. Large angle Bhabha scattering can also be used to determine the luminosity spectrum [2] at the International Linear Collider (ILC).

Because of the importance of Bhabha scattering, the theoretical description of this process is quite advanced. In particular, several Monte Carlo event generators have been developed to describe the Bhabha scattering process [3]; as a rule, these programs correctly reproduce the  $\mathcal{O}(\alpha)$  QED corrections and some parts of higher order corrections enhanced by large logarithms. Further progress in the theoretical description of Bhabha scattering requires the computation of  $\mathcal{O}(\alpha^2)$  next-to-next-to-leading (NNLO) QED corrections and the incorporation of those corrections into Monte Carlo event generators. The virtual two-loop corrections to  $e^+e^- \rightarrow e^+e^-$  scattering amplitude were calculated in ref. [14] in the approximation where lepton masses were set to zero. Unfortunately, this result is insufficient for the description of Bhabha scattering at flavor factories and the ILC where

isolated leptons rather than “QED jets” can be observed. Furthermore, all existing event generators work with massive electrons. To include the NNLO results into these codes, it is thus necessary to keep the electron mass also in fixed order calculations.

For all practical applications, having the logarithmic electron mass dependence of the NNLO corrections is sufficient. Corrections suppressed by powers of the electron mass are negligible, even at the smallest measured scattering angles. When performing the calculation of the cross section with a nonzero electron mass, one can thus expand the relevant diagrams in powers of the electron mass. Even so, the evaluation of the two-loop corrections remains a formidable task. Results for some of the necessary loop integrals were presented in refs. [4–10], but so far only the part of the cross section which involves closed electron loops has been evaluated [11–13].

It is possible to avoid the evaluation of the diagrams in the massive case. Instead, one can use the massless result of ref. [14] and restore the mass dependence in the logarithmic approximation. For corrections that do not involve closed fermion loops, to which we will refer to as “photonic”, this was shown in ref. [15] where the phenomenologically relevant result for these  $O(\alpha^2)$  corrections to the large-angle Bhabha scattering was first reported. However, the approach of ref. [15] is somewhat complicated since it requires to transform infrared and collinear  $1/\epsilon$  poles, inherent to massless amplitudes computed in dimensional regularization, to  $\ln \lambda$  and  $\ln m_e$  terms in the massive amplitude where the photon mass  $\lambda$  regularizes infrared divergences and the electron mass  $m_e$  regularizes collinear divergences.

In this paper we point out that a much simpler procedure for deriving massive amplitudes from massless ones exists since the two amplitudes are related by multiplicative renormalization factors. These renormalization factors can be deduced from the knowledge of massive and massless electron Dirac form-factors. The massive amplitude constructed along these lines has its infrared divergences regularized dimensionally and collinear divergences regularized by the electron mass. Beyond its simplicity, the advantage of this approach is that it can be directly applied to QCD whereas the method of ref. [15] relies on the photon mass as infrared regulator. In a recent paper, Mitov and Moch have obtained a similar relation between massless and massive amplitudes [16]. In fact, for photonic corrections our relation reduces to their result. In addition, our method allows us to also treat contributions involving massive fermion loops.

We apply our method to compute the NNLO QED corrections to large-angle Bhabha scattering; we include both photonic corrections and contributions from closed lepton loops. This calculation is a nontrivial application of our method. In addition, it provides an independent check of the computations of refs. [15, 11–13] with which we find complete agreement.<sup>1</sup> We also derive the NNLO contribution from loops with leptons heavier than the electron, i.e. muons and tau leptons, which was not available in the literature.

The paper is organized as follows. In the next section we present our notation and discuss the perturbative expansion of the large-angle Bhabha scattering cross-section. In section 3 we explain the factorization formula considering the electron Dirac form factor as an example. In section 4 we apply the factorization formula to compute the NNLO

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<sup>1</sup>Note that appendix B of ref. [12] contains a misprint [17].

QED corrections to Bhabha scattering. We conclude in section 5. Some useful formulas are collected in the appendix.

## 2. Notation

Consider the process  $e^+(p_1) + e^-(p_2) \rightarrow e^+(p_3) + e^-(p_4)$  for energies and scattering angles such that the absolute values of all kinematic invariants  $(p_1 + p_2)^2 = s$ ,  $(p_1 - p_3)^2 = t$  and  $(p_1 - p_4)^2 = u$  are much larger than the electron mass squared,  $s, |t|, |u| \gg m_e^2$ .

We compute the Bhabha scattering cross-section perturbatively in the on-shell scheme where the fine structure constant  $\alpha$  is defined through the photon propagator at zero momentum transfer. Neglecting corrections suppressed by powers of the electron mass, we write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \left( \frac{1-x+x^2}{x} \right)^2 \left[ 1 + \left( \frac{\alpha}{\pi} \right) \delta_1 + \left( \frac{\alpha}{\pi} \right)^2 \delta_2 + \mathcal{O}(\alpha^3) \right], \quad (2.1)$$

where  $x = (1 - \cos\theta)/2$  and  $\theta$  is the scattering angle. We take into account electron and muon contributions to photon vacuum polarization diagrams in the approximation  $s \gg m_\mu^2 \gg m_e^2$ . For phenomenological applications, the contribution of  $\tau$  leptons may be required; it can be obtained from the formulas below with the obvious modification  $m_\mu \rightarrow m_\tau$  provided that the high-energy condition  $s \gg m_\tau^2$  is valid.

The higher order corrections to the Bhabha scattering cross-section depend logarithmically on the mass of the electron. Also, in order to arrive at a physically meaningful result, we need to allow for soft radiation with the energy of each emitted photon below some value  $\omega_{\text{cut}} \ll m$ . The perturbative corrections depend logarithmically on  $\omega_{\text{cut}}$ . The corrections sensitive to soft and collinear physics are numerically enhanced relative to other corrections; it is therefore customary to separate out those corrections when presenting results for perturbative coefficients. The  $\mathcal{O}(\alpha)$  correction in eq. (2.1) is well-known [18]; in the limit of small electron mass it can be written as

$$\delta_1 = \left( 4L_{\text{soft}} + 3 + \frac{2}{3}N_f \right) \ln \left( \frac{s}{m_e^2} \right) + \delta_1^{(0)}, \quad (2.2)$$

where  $L_{\text{soft}} = \ln(2\omega_{\text{cut}}/\sqrt{s})$ . We have introduced a label  $N_f$  to distinguish corrections due to closed lepton loops. In particular, including electron and muon loops corresponds to  $N_f = N_e + N_\mu$  where  $N_i$  is the number of leptons of the  $i$ th flavor. To obtain numerical results one has to set  $N_e = 1$  and  $N_\mu = 1$ .

The part of the one-loop correction that is not enhanced by the logarithm of the electron mass to center-of-mass energy ratio reads

$$\begin{aligned} \delta_1^{(0)} = & \left[ -4 + 4 \ln \frac{x}{1-x} \right] L_{\text{soft}} - \frac{2N_\mu}{3} \ln \frac{m_\mu^2}{m_e^2} - 4 - \frac{2\pi^2}{3} \\ & - 2\text{Li}_2(x) + 2\text{Li}_2(1-x) - \frac{10N_f}{9} + f(x). \end{aligned} \quad (2.3)$$

The function  $f(x)$  is defined as

$$\begin{aligned}
 f(x) = (1-x+x^2)^{-2} & \left\{ \left( \frac{1}{3} - \frac{2}{3}x + \frac{9}{4}x^2 - \frac{13}{6}x^3 + \frac{4}{3}x^4 \right) \pi^2 + \left( 3 - 4x + \frac{9}{2}x^2 - \frac{3}{2}x^3 \right) \ln(x) \right. \\
 & + \left( \frac{3}{4}x - \frac{x^2}{4} - \frac{3}{4}x^3 + x^4 \right) \ln^2(x) + \left[ -\frac{1}{2}x - \frac{1}{2}x^3 + \left( 2 - 4x + \frac{7}{2}x^2 - x^3 \right) \ln(x) \right] \ln(1-x) \\
 & \left. + \left( -1 + \frac{5}{2}x - \frac{7}{2}x^2 + \frac{5}{2}x^3 - x^4 \right) \ln^2(1-x) + N_f \left( \frac{2}{3} - \frac{x}{3} \right) (1-x+x^2) \ln(x) \right\}. \quad (2.4)
 \end{aligned}$$

The second term in the perturbative expansion of the Bhabha scattering cross-section is enhanced by up to three powers of the logarithm of the electron mass. We write the NNLO correction as

$$\delta_2 = -\frac{N_f}{9} \ln^3\left(\frac{s}{m_e^2}\right) + \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)}, \quad (2.5)$$

where

$$\begin{aligned}
 \delta_2^{(2)} = 8L_{\text{soft}}^2 + \left( 12 + \frac{8}{3}N_f \right) L_{\text{soft}} + \frac{9}{2} + N_f \left( -\frac{1}{3} \ln \frac{x}{1-x} + \frac{55}{18} \right) \\
 + \frac{N_\mu}{3} \ln \frac{m_\mu^2}{m_e^2} + \frac{N_f^2}{3}; \quad (2.6)
 \end{aligned}$$

$$\begin{aligned}
 \delta_2^{(1)} = \left[ -16 + 16 \ln \frac{x}{1-x} \right] L_{\text{soft}}^2 + \left[ -28 - \frac{8}{3}\pi^2 + 12 \ln \frac{x}{1-x} - 8\text{Li}_2(x) + 8\text{Li}_2(1-x) \right. \\
 + N_f \left( \frac{8}{3} \ln \frac{x}{1-x} - \frac{64}{9} \right) + 4f(x) - \frac{8N_\mu}{3} \ln \frac{m_\mu^2}{m_e^2} \left. \right] L_{\text{soft}} - \frac{93}{8} - \frac{5\pi^2}{2} + 6\zeta(3) - 6\text{Li}_2(x) \\
 + 6\text{Li}_2(1-x) + 3f(x) + N_\mu \left( -\frac{1}{3} \ln^2 \frac{m_\mu^2}{m_e^2} + \left( \frac{2}{3} \ln \frac{x}{1-x} - \frac{37}{9} \right) \ln \frac{m_\mu^2}{m_e^2} \right) \\
 - N_f \left( \frac{8}{3} \text{Li}_2(x) + \frac{281}{27} \right) + N_f g(x) + N_f^2 \left( -\frac{10}{9} + \frac{(2-x)}{3(1-x+x^2)} \ln(x) \right) \\
 - \frac{2}{3} N_f N_\mu \ln \frac{m_\mu^2}{m_e^2}, \quad (2.7)
 \end{aligned}$$

and the function  $g(x)$  reads

$$\begin{aligned}
 g(x) = (1-x+x^2)^{-2} & \left\{ \left( \frac{2}{3}x^4 - \frac{5}{4}x^2 - \frac{1}{12}x^3 + \frac{17}{12}x - \frac{1}{3} \right) \ln^2(x) \right. \\
 & + \left( (1-2x) \left( \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{2}{3} \right) \ln(1-x) + \frac{37}{9} - \frac{56x}{9} + \frac{47x^2}{6} - \frac{67x^3}{18} + \frac{10x^4}{9} \right) \ln(x) \\
 & - \left( \frac{2}{3}x^2 - \frac{7}{6}x + \frac{2}{3} \right) (x^2 - x + 1) \ln^2(1-x) + \left( -\frac{10}{3}x^2 + \frac{31}{18}x^3 + \frac{31}{18}x - \frac{10}{9} - \frac{10}{9}x^4 \right) \times \\
 & \left. \times \ln(1-x) + \left( \frac{11}{12}x^2 + \frac{8}{9}x^4 - \frac{1}{9} + \frac{2}{9}x - \frac{23}{18}x^3 \right) \pi^2 \right\}. \quad (2.8)
 \end{aligned}$$

Except for the muon contributions, the terms  $\delta_2^{(2)}$  and  $\delta_2^{(1)}$  were computed in refs. [19–21] and the term  $\delta_2^{(0)}$  in eq. (2.5) was computed in refs. [15, 11–13]. We present the result of our computation of the term  $\delta_2^{(0)}$  below.

### 3. Mass factorization

We begin with the description of the method that we use to compute the NNLO QED corrections to the Bhabha scattering cross-section. The key to our approach is a factorization formula that relates massless and massive amplitudes for a given process; such a relationship between massless and massive amplitudes can be expected because of the well-known fact [22] that in physical gauges collinear divergences factorize into the wave function renormalization constants.

We will explain the factorization theorem using the Dirac form factor and then apply it to Bhabha scattering but we stress that the same relation applies to arbitrary scattering amplitudes in QED and QCD in the limit where all particle masses are much smaller than typical momentum transfers. The Soft-Collinear Effective Theory (SCET) [23–25] is an appropriate framework to analyze factorization properties of processes in this limit. This effective field theory is constructed by studying the perturbative expansion in QED or QCD and identifying those momentum regions in loop integrals that lead to singularities once the expansion of diagrams in small kinematic variables or masses is performed. These momentum modes are described by effective theory fields while the remaining contributions are integrated out and absorbed into the Wilson coefficients of the effective theory operators. The singularities relevant to our case arise when particles are soft or have momenta collinear to the external momenta. The effective theory description of the electric current requires two different collinear fields which interact via soft exchanges. To explain the structure of the result, we first consider the electron Dirac form factor in the limit where the electron momenta fulfill  $Q^2 = -(p_1 - p_2)^2 \gg p_1^2 \sim p_2^2 \sim m_e^2$ . At leading power in the effective theory, the vector current  $V_\mu = \bar{\psi}\gamma_\mu\psi$  takes the form

$$V_\mu = \int dsdt \tilde{C}_V(s, t) [\bar{\xi}_2 W_2](s\bar{n}_2) \gamma_\mu [W_1^\dagger \xi_1](t\bar{n}_1) + \mathcal{O}(Q^{-1}). \quad (3.1)$$

Here,  $n_1$  and  $n_2$  are the light-like reference vectors in the directions of  $p_1$  and  $p_2$ , respectively. The conjugate vectors  $\bar{n}_1$  and  $\bar{n}_2$  point in the opposite directions and fulfill  $\bar{n}_1 \cdot n_1 = \bar{n}_2 \cdot n_2 = 2$ . These reference vectors must be chosen such that  $\bar{n}_1 \cdot p_1 \bar{n}_2 \cdot p_2 = Q^2 + \mathcal{O}(m_e^2)$ . The collinear electron fields  $\xi_1$  and  $\xi_2$  are multiplied by light-like collinear Wilson lines

$$W_i(x) = \exp \left( ie \int_{-\infty}^0 ds \bar{n}_i \cdot A_{c,i}(x + s\bar{n}_i) \right). \quad (3.2)$$

The Fourier transform of the Wilson coefficient

$$C_V(Q^2) \equiv C_V(\bar{n}_1 \cdot p_1 \bar{n}_2 \cdot p_2) = \int dsdt \tilde{C}_V(s, t) e^{is\bar{n}_2 p_2 - it\bar{n}_1 p_1} \quad (3.3)$$

depends only on the hard scale  $Q^2$ , but is independent of the electron mass. To obtain  $C_V(Q^2)$ , we perform a matching calculation. The simplest way to do the matching is to use dimensional regularization and to calculate the on-shell form factor in the massless theory. In this case all loop diagrams in effective theory vanish and the bare Wilson coefficient  $C_V(Q^2)$  equals the on-shell Dirac form factor  $\tilde{F}(Q^2)$  of a massless electron.

The massless on-shell form factor has infrared divergences, which show up as poles in  $4-d = 2\epsilon$ . These poles correspond to ultra-violet divergences in the effective theory. Since the Wilson coefficients are independent of the small electron mass, the difference between massive and massless amplitudes can only arise from matrix elements of operators in the effective field theory. Off-shell Green's functions in the effective theory get contributions from soft interactions between external legs and collinear interactions in each sector. At leading power, soft photons have eikonal interactions with collinear fields; only the  $n_i \cdot A_s$  component of the soft photon field interacts with collinear electrons moving in the  $i$ th direction. These interactions can be removed by field redefinitions [24]

$$\xi_i(x) = S_i(x_-)\xi_i^{(0)}(x), \quad A_i^\mu(x) = S_i(x_-)A_i^{(0)\mu}S_i^\dagger(x_-), \quad (3.4)$$

where  $x_- = \frac{1}{2}(\bar{n}_i \cdot x)n_i^\mu$  and the soft Wilson line reads

$$S_i(x) = \exp\left(ie \int_{-\infty}^0 ds n_i \cdot A_{c,i}(x + sn_i)\right). \quad (3.5)$$

The current operator takes the form

$$V_\mu = \int ds dt \tilde{C}_V(s, t) [\bar{\xi}_2^{(0)} W_2^{(0)}](s\bar{n}_2) \gamma_\mu S_2^\dagger(0) S_1(0) [W_1^{(0)\dagger} \xi_1^{(0)}](t\bar{n}_1) + \mathcal{O}(Q^{-1}). \quad (3.6)$$

After the field redefinition eq. (3.4), there is no interaction between the different sectors of the theory. The matrix elements of the current operator factorize into collinear matrix elements for each direction, called jet-functions, and a soft function, which is given by the matrix element of the soft Wilson lines. This factorization of Green's functions into hard-, jet- and soft functions at large momentum transfers is a well known property of gauge theories [26–28].

For on-shell matrix elements the situation is especially simple because in the massless case jet and soft functions are trivial since effective theory loop diagrams are scaleless. For massive electrons, the collinear matrix elements are functions of the lepton masses, while the soft function also depends on the hard momentum  $Q$ . We therefore write

$$F(Q^2, \{m^2\}) = Z_J(\{m^2\})S(Q^2, \{m^2\})\tilde{F}(Q^2) + \mathcal{O}(m^2/Q^2). \quad (3.7)$$

The relation between the massive form factor  $F$  and the massless form factor  $\tilde{F}$  simplifies when only photonic corrections are considered. In that case higher order corrections to the soft matrix element in the massive theory are given by scaleless integrals and therefore vanish. This can be seen diagrammatically before performing the somewhat formal decoupling transformation in eq. (3.4). Hence, we conclude that for photonic corrections the soft function in eq. (3.7) is equal to one to all orders in QED perturbation theory and can be dropped from the right hand side of eq. (3.7). The relation between massless and massive form factors obtained in this way coincides with the relation discussed recently in ref. [16]. It is also consistent with the one-loop relation obtained in ref. [30].

However, a non-trivial soft function appears once vacuum polarization diagrams with massive particles are considered. In that case, it is easy to see that the soft momentum

contribution in the massive theory is not a scaleless integral and therefore does not vanish. Moreover, it exhibits non-trivial dependence on the hard scale  $Q$ . We write the soft matrix element  $S(Q^2, \{m^2\})$  as

$$S(Q^2, \{m^2\}) = 1 + \sum_{i=e,\mu} \delta S(Q^2, m_i^2), \quad (3.8)$$

where

$$\delta S(Q^2, m_i^2, N_i) = -N_i(4\pi\alpha_0)^2 \int \frac{d^d k}{(2\pi)^d} \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)k^2} i\Pi(k^2, m_i^2). \quad (3.9)$$

In eq. (3.9)  $\alpha_0$  stands for the bare QED coupling constant. The vacuum polarization function  $\Pi(k^2)$  in eq. (3.9) is defined as

$$i\Pi(k^2, m_i^2) = \frac{(-1)}{(d-1)k^2} \int \frac{d^d l}{(2\pi)^d} \text{Tr} \left[ \gamma_\alpha \frac{1}{\not{l} - m_i} \gamma^\alpha \frac{1}{\not{l} + \not{k} - m_i} \right]. \quad (3.10)$$

The easiest way to compute the soft matrix element is to employ a dispersive representation for the vacuum polarization function  $\Pi(k^2)$ . Evaluating the integral in eq. (3.9) in  $d = 4 - 2\epsilon$  dimensions, we obtain

$$\delta S(Q^2, m_i^2, N_i) = N_i a_0^2 m_i^{-4\epsilon} \ln \left( \frac{Q^2}{m_e^2} \right) \left( -\frac{1}{12\epsilon^2} + \frac{5}{36\epsilon} - \frac{7}{27} - \frac{\pi^2}{72} + \mathcal{O}(\epsilon) \right), \quad (3.11)$$

where  $a_0 = \alpha_0/\pi e^{-\gamma\epsilon}(4\pi)^\epsilon$  and  $\gamma$  is the Euler constant. Eq. (3.7) provides a relation between closed lepton loop contributions to massive and massless form factors and allows a derivation of  $N_{e,\mu}$ -dependent  $\mathcal{O}(\alpha^2)$  corrections to Bhabha scattering from the massless results of ref. [14].

To determine the square of the jet-function  $Z_J$  to NNLO in QED, we use eq. (3.7) and divide the ratio of dimensionally regularized massive and massless form factors by the soft matrix element eq. (3.9). The expansion of the massive vector form factor in the limit  $Q^2 \gg m_e^2$  through  $\mathcal{O}(\alpha^2)$  can be found in refs. [31–33].<sup>2</sup> The massless result can be obtained from refs. [34, 35]. We express the jet function through the bare QED coupling constant  $\alpha_0$ . We find

$$\begin{aligned} Z_J = & 1 + a_0 m_e^{-2\epsilon} \left[ \frac{1}{2\epsilon^2} + \frac{1}{4\epsilon} + \frac{\pi^2}{24} + 1 + \epsilon \left( 2 + \frac{\pi^2}{48} - \frac{\zeta(3)}{6} \right) + \epsilon^2 \left( 4 - \frac{\zeta(3)}{12} + \frac{\pi^4}{320} + \frac{\pi^2}{12} \right) \right] \\ & + a_0^2 m_e^{-4\epsilon} \left[ \frac{1}{8\epsilon^4} + \frac{1}{\epsilon^3} \left( \frac{1}{8} - \frac{N_f}{24} \right) + \frac{1}{\epsilon^2} \left( \frac{17}{32} + \frac{\pi^2}{48} - \frac{N_f}{36} \right) + \frac{1}{\epsilon} \left( \frac{83}{64} - \frac{\pi^2}{24} + \frac{2\zeta(3)}{3} \right) \right. \\ & \left. - N_f \left( \frac{209}{432} + \frac{5\pi^2}{144} \right) - \frac{N_\mu}{6} \ln \frac{m_\mu^2}{m_e^2} \right] + \frac{561}{128} + \frac{61\pi^2}{192} - \frac{11}{24}\zeta(3) - \frac{\pi^2}{2} \ln(2) - \frac{77\pi^4}{2880} \end{aligned}$$

<sup>2</sup>We need the  $\mathcal{O}(\epsilon^2)$  terms in the one-loop contribution to the massive form factor which are not provided in ref. [31]. However, it is simple to compute these terms and the corresponding result is given in appendix A. Note also that with the normalization adopted in eq. (24) of ref. [31] the  $\overline{\text{MS}}$  result for the contribution of massive quark vacuum polarization to the heavy quark form factor coincides with the result in the on-shell scheme.



$$\begin{aligned}
 & +N_f \left( \frac{3379}{2592} - \frac{19\pi^2}{216} + \frac{\zeta(3)}{36} \right) + N_\mu \left( \frac{1}{36} \ln^3 \frac{m_\mu^2}{m_e^2} + \frac{25}{72} \ln^2 \frac{m_\mu^2}{m_e^2} + \left( \frac{193}{216} + \frac{\pi^2}{18} \right) \ln \frac{m_\mu^2}{m_e^2} \right. \\
 & \left. - \frac{1241}{1296} + \frac{7\pi^2}{54} - \frac{\zeta(3)}{3} \right) \Big] + \mathcal{O}(\alpha\epsilon^3, \alpha^2\epsilon). \tag{3.12}
 \end{aligned}$$

Setting  $N_f = N_\mu = 0$ , we find the agreement with the result of ref. [16]. The independence of the jet function  $Z_J$  of the hard scale  $Q^2$  is an explicit demonstration of the mass factorization to two-loop order.

Before turning to Bhabha scattering, we want to address a subtlety concerning eq. (3.7). This relation relies on the assumption that only hard, collinear and soft momentum modes are relevant in the effective theory computation. However, as was explicitly shown in ref. [29], this assumption is invalid for some diagrams that contribute to the form factor. A particular example discussed in ref. [29] is the contribution of so-called ultra-collinear modes to a two-loop planar vertex diagram. Let us stress that the relevance of these modes for the full form factor computation would invalidate the factorization formula eq. (3.7) since this mode induces additional dependence on the hard scale  $Q$  that is not associated with the hard or soft region. It is therefore gratifying to observe that the ultra-collinear modes discussed in ref. [29] are not relevant for the form factor since their contributions cancel out. For example, at the two-loop level, the ultra-collinear contribution to a planar vertex diagram cancels exactly against a similar contribution to the non-planar vertex diagram making the full form factor independent of it.

#### 4. Bhabha scattering

We can now apply the factorization formula, established for the electron Dirac form factor in the previous section, to Bhabha scattering. To get the scattering amplitude  $\mathcal{M}$  in which the electron mass is used as a regulator of the collinear singularities, we only need to multiply the massless amplitude  $\tilde{\mathcal{M}}$  by the square root of the jet function  $Z_J^{1/2}$  for each electron and positron leg and by the product of soft functions that account for soft exchanges in  $s, t$  and  $u$  channels. To obtain the soft matrix element, we may use eq. (3.11). While eq. (3.11) is relevant for the space-like electron form factor, we need to generalize it to describe soft exchanges in the  $s$ - and  $u$ -channel. We write

$$\mathcal{M}(\{p_i\}, \{m^2\}) = Z_J^2(\{m^2\}) \tilde{\mathcal{M}}(\{p_i\}) S(s, t, u) + \mathcal{O}(m^2/Q^2), \tag{4.1}$$

where the soft function  $S(s, t, u)$  is given by

$$S(s, t, u) = \left( 1 + 2 \sum_{Q^2} \sum_{i=e,\mu} \delta S(Q^2, m_i^2, \lambda_{Q^2} N_i) \right) \tag{4.2}$$

and the sum goes over  $Q^2 = -s, -t, -u$  with  $\lambda_s = \lambda_t = 1$  and  $\lambda_u = -1$ . The change  $N_i \rightarrow -N_i$  shown in eq. (4.2) accounts for the required change in the overall sign in  $\delta S$  in the  $u$ -channel.

We can now use the two-loop result for the Bhabha scattering amplitude computed in the massless limit [14] and employ eqs. (3.12), (4.1) and (4.2) to obtain the scattering amplitude in which collinear divergences are regularized by the electron mass and infrared divergences are regularized dimensionally. With the massive scattering amplitude at hand, the computation of the two-loop QED corrections to the large-angle Bhabha scattering cross-section becomes straightforward; what we need in addition is the cross-sections for the inelastic processes  $e^+e^- \rightarrow e^+e^- + n\gamma$ , with  $n = 1$  and  $n = 2$ , in the soft photon approximation. We write the perturbative expansion of the Bhabha scattering cross-section as

$$\frac{d\sigma}{d\Omega} = \exp\left(\frac{\alpha}{\pi}F_{\text{soft}}\right) Z_J^4 |S|^2 \frac{\bar{\alpha}^2}{s} \left( d\sigma_0 + \frac{\bar{\alpha}}{\pi}d\sigma_1^v + \left(\frac{\bar{\alpha}}{\pi}\right)^2 (d\sigma_{1\times 1}^v + d\sigma_2^v) + \mathcal{O}(\alpha^3) \right). \quad (4.3)$$

In this formula,  $Z_J$  is the square of the jet-function given in eq. (3.12);  $F_{\text{soft}}$  describes soft photon radiation which, in case of QED, is known to factorize and exponentiate. Up to an overall factor  $\bar{\alpha}^2/s$  the quantity  $d\sigma_0$  is the tree level cross section in  $d$ -dimensions and  $d\sigma_1^v$  denotes the one-loop virtual contributions. At two loops, there are two types of virtual corrections: the quantity  $d\sigma_2^v$  contains the interference of the two-loop amplitude with the tree level amplitude, while  $d\sigma_{1\times 1}^v$  describes the interference of one-loop amplitude with itself. These contributions are to be computed with massless leptons. The massive result for the virtual corrections is then obtained by multiplying the massless result with  $Z_J^4 |S|^2$ . The massless cross-sections  $d\sigma_1^v$  and  $d\sigma_2^v$  can be found in ref. [14], while  $d\sigma_{1\times 1}^v$  can be obtained from ref. [36], as we explain below. Note also that results of refs. [14, 36] are written through the QED coupling constant  $\bar{\alpha}$  renormalized in the  $\overline{\text{MS}}$  scheme. It is for this reason that  $\bar{\alpha}$  appears in eq. (4.3).

In ref. [36] the interference of the one-loop amplitude with itself was obtained for quark-quark scattering in QCD. To extract the QED piece relevant for Bhabha scattering from these results, we need to analyze the color algebra; such an analysis shows that  $d\sigma_{1\times 1}^v$  can be obtained from the computation of ref. [36] by taking the  $N \rightarrow 0$  limit of the QCD result, where  $N$  is the number of colors, and subtracting from it suitably weighted products of the one-loop Bhabha scattering cross-section  $d\sigma_1^v$  and the electron Dirac form factor in the massless approximation. Note that the divergent terms in  $d\sigma_{1\times 1}^v$  can be obtained from Catani's decomposition [37] of the one-loop scattering amplitude for  $e^+e^- \rightarrow e^+e^-$  and the  $\mathcal{O}(\alpha)$  correction to the Bhabha scattering cross-section in dimensional regularization derived in ref. [14]. For this reason we only present the finite part of  $d\sigma_{1\times 1}^v$ ; it is given in the appendix B.

Finally, the description of soft radiation in eq. (4.3) is encapsulated in the function  $F_{\text{soft}}$ ; we require this function for massive electron-positron scattering. Since  $F_{\text{soft}}$  describes the emission of a real photon, it is simplest to evaluate it in the on-shell scheme. In doing so, the effects of vacuum polarization contributions are absorbed into the coupling constant and do not need to be evaluated explicitly. The function  $F_{\text{soft}}$  is determined by the integral

$$F_{\text{soft}} = -4\pi^2 \int_{k_0 \leq \omega_{\text{cut}}} \frac{d^d k}{(2\pi)^{d-1} 2k_0} J_\mu J^\mu, \quad (4.4)$$

where the soft current  $J^\mu$  is given by

$$J^\mu = \sum_i q_i \lambda_i \frac{p_i^\mu}{p_i \cdot k}, \quad (4.5)$$

$q_i$  is the charge of the particle  $i$  in units of the positron charge and  $\lambda = \pm 1$  for the incoming(outgoing) particle, respectively. Writing

$$J_\mu J^\mu = \sum_{i \neq j} \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} q_i q_j \lambda_i \lambda_j + \sum_i \frac{m_e^2}{(p_i \cdot k)^2}, \quad (4.6)$$

we observe that two types of integrals are required for the evaluation of  $F_{\text{soft}}$ . We give expressions for these integrals below neglecting all the terms that are suppressed by powers of the electron mass.

The first integral depends on the relative momenta of two charged particles. We find

$$\begin{aligned} I_{ij} &= \int_{k_0 \leq \omega_{\text{cut}}} \frac{d^{d-1}k}{(2\pi)^{d-1} 2k_0} \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} \\ &= -\mathcal{N}_\epsilon \frac{(2\omega_{\text{cut}})^{-2\epsilon}}{2\epsilon} \left[ \ln \frac{1}{1+x_{ij}} - L_m + \epsilon \left( \frac{1}{2} L_m^2 + \text{Li}_2(-x_{ij}) + \frac{\pi^2}{3} \right) \right. \\ &\quad \left. + \epsilon^2 \left( -\frac{L_m^3}{6} - \frac{\pi^2}{3} L_m + 2\zeta(3) + \text{Li}_3(-x_{ij}) \right) \right], \end{aligned} \quad (4.7)$$

where  $x_{ij} = (1 + \cos \theta_{ij}) / (1 - \cos \theta_{ij})$ ,  $\theta_{ij}$  is the relative angle between the three-momenta of the particles  $i$  and  $j$  and the normalization factor  $\mathcal{N}_\epsilon$  reads

$$\mathcal{N}_\epsilon = \frac{\Gamma(1-\epsilon)}{4\pi^2 (4\pi)^{-\epsilon} \Gamma(1-2\epsilon)}. \quad (4.8)$$

Also, we introduced  $L_m = \ln(m_e^2/s)$  to denote the collinear logarithm. The second integral required to describe the soft radiation reads

$$I_s = \int_{k_0 \leq \omega_{\text{cut}}} \frac{d^{d-1}k}{(2\pi)^{d-1} 2k_0} \frac{m_e^2}{(p_i \cdot k)^2} = -\mathcal{N}_\epsilon \frac{(2\omega_{\text{cut}})^{-2\epsilon}}{2\epsilon} \left[ 1 - \epsilon \ln \frac{m_e^2}{s} + \frac{\epsilon^2}{2} \ln^2 \frac{m_e^2}{s} \right], \quad (4.9)$$

where we used the on-shell condition  $p_i^2 = m_e^2$ .

We now have everything in place to calculate the NNLO QED corrections to Bhabha scattering. We substitute all the necessary ingredients into eq. (4.3). To combine the different pieces, it is simplest to first express the on-shell coupling constant appearing in the soft radiation exponential in eq. (4.3) through the  $\overline{\text{MS}}$  coupling constant. In  $d$ -dimensions, the relevant relation reads

$$\alpha = \bar{\alpha}(\mu) \mu^{2\epsilon} \left\{ 1 - \frac{2\bar{\alpha}(\mu)}{3\pi} \sum_{f=e,\mu} \left[ \ln \frac{\mu}{m_f} + \epsilon \left( \ln^2 \frac{\mu}{m_f} + \frac{\pi^2}{24} \right) + \mathcal{O}(\epsilon^2) \right] \right\}. \quad (4.10)$$

After adding the real and virtual parts, all infrared divergences cancel and we can use the four-dimensional relation between the  $\overline{\text{MS}}$  and the on-shell fine structure constants

$$\begin{aligned} \bar{\alpha}(\mu) = \alpha \left\{ 1 + \left(\frac{\alpha}{\pi}\right) \left( \frac{2N_f}{3} \ln \frac{\mu}{m_e} - \frac{2N_\mu}{3} \ln \frac{m_\mu}{m_e} \right) \right. \\ \left. + \left(\frac{\alpha}{\pi}\right)^2 \left[ \frac{4N_f^2}{9} \ln^2 \frac{\mu}{m_e} + \frac{N_f}{2} \ln \frac{\mu}{m_e} + \frac{15N_f}{16} \right. \right. \\ \left. \left. - \frac{8N_f N_\mu}{9} \ln \frac{\mu}{m_e} \ln \frac{m_\mu}{m_e} - \frac{N_\mu}{2} \ln \frac{m_\mu}{m_e} + \frac{4N_\mu^2}{9} \ln^2 \frac{m_\mu}{m_e} \right] \right\}, \end{aligned} \quad (4.11)$$

to obtain the expansion of the Bhabha scattering cross section through NNLO in terms of the on-shell QED coupling constant. Upon doing so, we reproduce the formulas for radiative corrections to Bhabha scattering shown in section 2 and obtain an explicit expression for  $\delta_2^{(0)}$ , which is presented below. We write  $\delta_2^{(0)}$  in the following way

$$\delta_2^{(0)} = \delta_2^{(0,1)} + N_f \delta_2^{(0,2)} + N_\mu \delta_2^{(0,3)} + N_f^2 \delta_2^{(0,4)} + N_f N_\mu \delta_2^{(0,5)} + \frac{N_\mu^2}{3} \ln^2 \frac{m_\mu^2}{m_e^2}, \quad (4.12)$$

where we separate photonic corrections and the corrections caused by closed lepton loops. We now present the results for these terms separately. For the photonic corrections we find

$$\begin{aligned} \delta_2^{(0,1)} = 8\mathcal{L}_{\text{soft}}^2 + (1-x+x^2)^{-2} \mathcal{L}_{\text{soft}} \left( -x^2(4x^2+5-6x)\pi^2 - x(3-3x^2-x+4x^3) \ln^2(x) \right. \\ \left. + [2x^2(4x^2+5-6x) \ln(1-x) - 12+16x-18x^2+6x^3] \ln(x) + 2x(x^2+1) \ln(1-x) \right. \\ \left. + 2(x^2-x+1)(2x^2-3x+2) \ln^2(1-x) + 16(x^2-x+1)^2 (1+\text{Li}_2(x)) \right) \\ \left. + 8\text{Li}_2(x)^2 + \frac{27}{2} - 2\pi^2 \ln(2) + (1-x+x^2)^{-2} \Delta_2^{(0,1)} \right), \end{aligned} \quad (4.13)$$

where  $\mathcal{L}_{\text{soft}} = (1 - \ln(x/(1-x))) L_{\text{soft}}$ , and

$$\begin{aligned} \Delta_2^{(0,1)} = & + \left( \frac{31}{480}x^4 - \frac{8}{45} + \frac{37}{90}x^2 - \frac{7}{72}x - \frac{47}{180}x^3 \right) \pi^4 + \left[ \frac{1}{48}x(35x^3 - 2x^2 + 20x + 24) \ln^2(x) \right. \\ & + \left( \left( -\frac{35}{24}x^4 + \frac{8}{3}x^3 - \frac{11}{12}x^2 - \frac{5}{2}x + \frac{8}{3} \right) \ln(1-x) - \frac{15}{8}x^3 + \frac{11}{12}x^2 + \frac{23}{12}x - \frac{1}{2} \right) \ln(x) \\ & + \left( \frac{5}{48}x^4 + \frac{1}{12}x^3 - \frac{7}{3}x^2 + 3x - \frac{49}{24} \right) \ln^2(1-x) + \frac{1}{24}x(43x^2 - 74x + 24) \ln(1-x) \\ & \left. - \frac{3}{4}x^2 + \frac{17}{8} + \frac{83}{24}x^3 - \frac{19}{8}x^4 - \frac{61}{24}x \right] \pi^2 + \frac{1}{96}(43x^3 - 8x^2 + 5x + 14)x \ln^4(x) \\ & + \left( \left( -\frac{43}{24}x^4 + \frac{7}{6}x^3 + \frac{1}{2}x^2 - \frac{17}{12}x + \frac{2}{3} \right) \ln(1-x) - \frac{1}{24}(16x^2 + 30x - 67)x \right) \ln^3(x) \\ & + \left( \left( \frac{19}{16}x^4 - \frac{29}{8}x^3 + \frac{7}{8}x^2 + \frac{39}{8}x - \frac{9}{2} \right) \ln^2(1-x) + \frac{1}{8}(9x^2 - 2x - 24)x \ln(1-x) \right. \\ & \left. - \frac{9}{2}x^4 + \frac{29}{8}x^3 + \frac{17}{8}x^2 - \frac{43}{8}x + \frac{9}{2} \right) \ln^2(x) + \left( \left( -\frac{1}{8}x^4 + \frac{1}{3}x^3 + \frac{11}{3}x^2 - \frac{37}{6}x + 4 \right) \ln^3(1-x) \right) \end{aligned}$$

$$\begin{aligned}
 & + \left( \frac{13}{8}x^3 - \frac{11}{2}x^2 + \frac{27}{4}x - 3 \right) \ln^2(1-x) + \frac{1}{4}x(51x - 22 + 36x^3 - 64x^2) \ln(1-x) \\
 & + (12 - 12x + 8x^2 - x^3)\zeta(3) - \frac{279}{16}x^2 + \frac{231}{16}x + \frac{93}{16}x^3 - \frac{93}{8} \ln(x) \\
 & + \left( \frac{1}{32} - \frac{3}{4}x + \frac{71}{48}x^2 - \frac{29}{24}x^3 + \frac{9}{32}x^4 \right) \ln^4(1-x) + \frac{1}{24}(9x^2 + 4x + 9)x \ln^3(1-x) \\
 & + \left( \frac{45}{4}x^2 - 6x + \frac{7}{2}x^4 - 6x^3 + \frac{7}{2} \right) \ln^2(1-x) \\
 & + ((-4x^3 - 6 + 6x - x^2)\zeta(3) + 3x(x^2 + 1)) \ln(1-x) \\
 & + \left( -9 + \frac{43}{2}x - 34x^2 + 22x^3 - 9x^4 \right) \zeta(3) + \left[ \left( -\frac{17}{4}x - 3x^4 + 2 + \frac{9}{4}x^2 + \frac{7}{4}x^3 \right) \ln^2(x) \right. \\
 & + \left. \left( (-6 + 7x + \frac{15}{2}x^2 + 8x^4 - 14x^3) \ln(1-x) + \frac{27}{2}x - 13x^2 + 4x^3 - 12 \right) \ln(x) \right. \\
 & + \left. \frac{1}{2}(x^2 - 4x + 7)(2x^2 - 3x + 2) \ln^2(1-x) + \frac{1}{2}x(5 + 3x^2) \ln(1-x) \right. \\
 & \left. - 32x - 32x^3 + 16x^4 + 16 + 48x^2 + \pi^2 \left( \frac{31}{6}x^3 + \frac{5}{3}(1-x) - \frac{15}{4}x^2 - \frac{8}{3}x^4 \right) \right] \text{Li}_2(x) \\
 & + \left( (-6 + 5x + 3x^2 - 5x^3) \ln(x) + 2(1-x^2)(3x^2 - 5x + 3) \ln(1-x) + \frac{x}{2}(1-x^2) \right) \text{Li}_3(1-x) \\
 & + \left( (-4 - x + x^2 + 2x^3 - 2x^4) \ln(x) + (x^2 + 6 + 4x^3 - 6x) \ln(1-x) + \frac{x}{2}(4x^2 - 10x + 5) \right) \times \\
 & \times \text{Li}_3(x) + \left( -6 + 4x + \frac{9}{2}x^2 - 7x^3 \right) \text{Li}_4 \left( \frac{x}{x-1} \right) + \frac{x}{2} (12x^3 + 14 - 9x - 8x^2) \text{Li}_4(1-x) \\
 & - \frac{1}{2}(1-x^2)(4x-1)(x-4)\text{Li}_4(x). \tag{4.14}
 \end{aligned}$$

The contribution of diagrams with a single electron or muon vacuum polarization insertion is described by  $\delta_2^{(0,2)}$ ,  $\delta_2^{(0,3)}$ . We obtain

$$\delta_2^{(0,2)} = \mathcal{L}_{\text{soft}} \left( \frac{40}{9} + \frac{4(x-2)}{3(1-x+x^2)} \ln(x) \right) + \frac{40}{9} \text{Li}_2(x) + \frac{1967}{108} + (1-x+x^2)^{-2} \Delta_2^{(0,2)}, \tag{4.15}$$

where

$$\begin{aligned}
 \Delta_2^{(0,2)} & = \left( \frac{2}{3}x^2 - x + \frac{4}{3} \right) (x^2 - x + 1) \text{Li}_3(x) + \frac{2}{3}(1-x^2)(x^2 - x + 1) \text{Li}_3(1-x) \\
 & + \left( -(x^2 - x + 1) \left( \frac{2}{3}x^2 - \frac{7}{3}x + 4 \right) \ln(x) + \frac{2}{3}(x^2 - x + 1)(1-x^2) \ln(1-x) \right) \text{Li}_2(x) \\
 & + \left( -\frac{1}{18}x^3 - \frac{11}{18}x^2 + \frac{1}{9}x^4 + \frac{31}{36}x - \frac{1}{9} \right) \ln^3(x) + \left( \left( \frac{4}{3}x^2 - \frac{1}{3}x^4 + \frac{1}{3}x - \frac{1}{3} + x^3 \right) \ln(1-x) \right. \\
 & \left. - \frac{10}{9}x^4 + \frac{14}{3}x^2 - \frac{4}{9}x^3 - \frac{46}{9}x + \frac{55}{18} \right) \ln^2(x) + \left( -x \left( \frac{7}{12}x - \frac{1}{3} + \frac{x^3}{3} - \frac{x^2}{2} \right) \ln^2(1-x) \right. \\
 & \left. + \left( \frac{1}{2}x^2 - \frac{10}{9} - \frac{25}{9}x^3 + \frac{37}{18}x + \frac{20}{9}x^4 \right) \ln(1-x) + \left( -\frac{1}{36}x^2 + \frac{1}{3}x^4 - \frac{1}{9} + \frac{2}{3}x - \frac{5}{9}x^3 \right) \pi^2 \right. \\
 & \left. - \frac{337}{18}x^2 + \frac{449}{54}x^3 - \frac{281}{27} + \frac{418}{27}x - \frac{56}{27}x^4 \right) \ln(x) - \frac{1}{9}(x^2 - x + 1)(1-x)^2 \ln^3(1-x)
 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{10}{9} + \frac{9}{2}x^2 + \frac{10}{9}x^4 - \frac{29}{9}x - \frac{29}{9}x^3 \right) \ln^2(1-x) + \left( \left( -\frac{16}{9}x^2 - \frac{2}{9}x^4 - \frac{4}{9} + \frac{11}{9}x + x^3 \right) \pi^2 \right. \\
& + \frac{56}{9}x^2 - \frac{161}{54}x^3 + \frac{56}{27} - \frac{161}{54}x + \frac{56}{27}x^4 \left. \right) \ln(1-x) + \left( \frac{47}{12}x^3 - \frac{7}{3}x^4 - \frac{1}{6} + \frac{11}{12}x - \frac{71}{18}x^2 \right) \pi^2 \\
& + (x^2 - x + 1) \left( 2x^2 - \frac{5x}{3} + \frac{4}{3} \right) \zeta(3), \tag{4.16}
\end{aligned}$$

and

$$\begin{aligned}
\delta_2^{(0,3)} &= \frac{8}{3} \ln \frac{m_\mu^2}{m_e^2} \mathcal{L}_{\text{soft}} + \frac{1}{9} \ln^3 \frac{m_\mu^2}{m_e^2} + \left( \frac{19}{18} - \frac{1}{3} \ln \frac{x}{1-x} \right) \ln^2 \frac{m_\mu^2}{m_e^2} \\
& + \left( \frac{8}{3} \text{Li}_2(x) + \frac{191}{27} + (1-x+x^2)^{-2} \Delta_2^{(0,3)} \right) \ln \frac{m_\mu^2}{m_e^2} + \frac{14\pi^2}{27} - \frac{4}{3} \zeta(3) - \frac{1241}{324}, \tag{4.17}
\end{aligned}$$

where

$$\begin{aligned}
\Delta_2^{(0,3)} &= (1-x+x^2) \left( \frac{2}{3}x^2 - \frac{7}{6}x + \frac{2}{3} \right) \ln^2(1-x) + \left( (2x-1) \left( \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{2}{3} \right) \ln(x) \right. \\
& + \frac{10}{9} + \frac{10}{9}x^4 - \frac{31}{18}x - \frac{31}{18}x^3 + \frac{10}{3}x^2 \left. \right) \ln(1-x) + \left( \frac{1}{12}x^3 - \frac{2}{3}x^4 - \frac{17}{12}x + \frac{5}{4}x^2 + \frac{1}{3} \right) \ln^2(x) \\
& + \left( \frac{37}{9} - \frac{10}{9}x^4 + \frac{56}{9}x + \frac{67}{18}x^3 - \frac{47}{6}x^2 \right) \ln(x) + \left( \frac{23}{18}x^3 - \frac{8}{9}x^4 + \frac{1}{9} - \frac{11}{12}x^2 - \frac{2}{9}x \right) \pi^2. \tag{4.18}
\end{aligned}$$

The corrections with two insertions of the closed lepton loop read

$$\delta_2^{(0,4)} = \frac{25}{27} + (1-x+x^2)^{-2} \Delta_2^{(0,4)}, \tag{4.19}$$

where

$$\begin{aligned}
\Delta_2^{(0,4)} &= \left( \frac{1}{3} - \frac{x^3}{9} + \frac{7x^2}{18} - \frac{4x}{9} \right) \ln^2(x) - \frac{5}{9}(x^2-x+1)(2-x) \ln(x) \\
& + x \left( \frac{1}{9} - \frac{5x}{18} - \frac{x^3}{9} + \frac{2x^2}{9} \right) \pi^2. \tag{4.20}
\end{aligned}$$

and

$$\delta_2^{(0,5)} = \left( \frac{10}{9} + \frac{(x-2)}{3(1-x+x^2)} \ln x \right) \ln \frac{m_\mu^2}{m_e^2}. \tag{4.21}$$

Because the above expressions are rather lengthy, we have included them in electronic form in our submission to the arXiv.

## 5. Conclusion

We presented a novel relation between massive and massless scattering amplitudes in QED valid in the limit when all kinematic invariants are large compared to masses of particles that participate in the scattering process; for quenched QED, our result agrees with a similar relation between massive and massless scattering amplitudes discussed recently in [16]. We used this relation to derive the NNLO QED corrections for Bhabha scattering

confirming earlier results on photonic and electron loop corrections of ref. [15, 11–13]. We also obtained NNLO contributions to Bhabha scattering due to muon and tau vacuum polarization loops that were not available in the literature.

In variance with the approach discussed in ref. [15], our method is directly applicable to QCD. A potentially interesting application is the computation of the NNLO QCD virtual corrections to heavy (e.g.  $b$ ) quark production at moderate to large momentum transfers at the Tevatron and the LHC using the two-loop matrix elements for the  $gg \rightarrow q\bar{q}$  process computed in ref. [38] for massless quarks.

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**Note added.** While we were finalizing our paper, ref. [39] appeared in which the muon-loop contribution to Bhabha scattering was evaluated. After the authors of ref. [39] corrected one of the master integrals their result agrees with ours.

## A. The one-loop massive Dirac form factor

The one-loop massive Dirac space-like form factor, in the limit  $Q^2 \gg m_e^2$ , reads

$$\begin{aligned}
 F_1 = 1 + \frac{\bar{\alpha}(m_e)}{2\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} & \left[ \frac{L-1}{\epsilon} - \frac{L^2}{2} + \frac{3L}{2} - 2 + \frac{\pi^2}{6} + \epsilon \left( \frac{L^3}{6} - \frac{3L^2}{4} + \left( 4 - \frac{\pi^2}{6} \right) L + 2\zeta(3) \right. \right. \\
 & \left. \left. + \frac{\pi^2}{4} - 4 \right) + \epsilon^2 \left( -\frac{L^4}{24} + \frac{L^3}{4} + \left( \frac{\pi^2}{12} - 2 \right) L^2 + \left( 8 - \frac{\pi^2}{4} - 2\zeta(3) \right) L \right. \right. \\
 & \left. \left. + 3\zeta(3) + \frac{2\pi^2}{3} - 8 + \frac{\pi^4}{40} \right) \right], \tag{A.1}
 \end{aligned}$$

where  $L = \ln(Q^2/m_e^2)$  and  $\bar{\alpha}(m)$  is the  $\overline{\text{MS}}$  fine structure constant evaluated at the scale  $m$ .

## B. NNLO results

Here, we give the finite part of the  $\mathcal{O}(\alpha^2)$  correction to the crosssection due to the interference of the *photonic* one-loop amplitude with itself. The divergent part can be obtained using Catani’s results [37].

$$\begin{aligned}
 s^{2\epsilon} d\sigma_{1\times 1}^v|_{\text{finite}} = & \frac{(8x^4 - 13x^3 + 13x^2 - 7x + 4)}{2x^2} \text{Li}_4\left(\frac{x}{x-1}\right) \\
 & - \frac{(8 - 13x + 13x^2 - 7x^3 + 4x^4)}{2x^2} \text{Li}_4(1-x) - \frac{(1-x^2)(2x^2 - 3x + 2)}{x^2} \text{Li}_4(x)
 \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \frac{(-x^2 - 9x^3 + 8x^4 + 9x - 4)}{2x^2} \ln(x) + \frac{(1-x^2)(2x^2 - 3x + 2)}{x^2} \ln(1-x) \right. \\
 & + \left. \frac{3(1-x^2)(2x^2 - 3x + 2)}{2x^2} \right\} \text{Li}_3(1-x) + \left\{ -\frac{(13x^2 - 7x^3 + 4x^4 - 13x + 8)}{2x^2} \ln(x) \right. \\
 & + \left. \frac{(4 - 3x - x^2 + 3x^3)}{2x^2} \ln(1-x) + \frac{(12x^4 - 9x^3 + 24 - 27x + 26x^2)}{4x^2} \right\} \text{Li}_3(x) \\
 & + \left\{ \frac{3(13x^2 - 7x^3 + 4x^4 - 13x + 8)}{4x^2} \ln^2(x) + \left( \frac{(-13x^2 + x^3 + 19x - 12)}{2x^2} \ln(1-x) \right. \right. \\
 & \left. \left. - \frac{(12x^4 - 9x^3 + 24 - 27x + 26x^2)}{4x^2} \right) \ln(x) + \frac{(1-x^2)(2x^2 - 3x + 2)}{2x^2} \ln^2(1-x) \right. \\
 & + \left. \frac{3(1-x^2)(2x^2 - 3x + 2)}{2x^2} \ln(1-x) - \frac{(13x^2 - 7x + 4 - 13x^3 + 8x^4)\pi^2}{4x^2} \right\} \text{Li}_2(x) \\
 & - \frac{(45x^4 + 26x^3 - 241x^2 + 230x - 64)}{96x^2} \ln^4(x) + \left( \frac{(45x^4 - 38x^3 - 8x^2 + 28x + 16)}{24x^2} \ln(1-x) \right. \\
 & + \left. \frac{(12x^4 + 15x^3 - 93x^2 + 169x - 96)}{24x^2} \right) \ln^3(x) + \left( -\frac{(12x^4 - 16x^3 + 13x^2 + 27x - 24)}{8x^2} \ln(1-x) \right. \\
 & + \left. \frac{(11x^4 - 10x^3 - 38x^2 + 62x - 40)}{16x^2} \ln^2(1-x) + \frac{(19x^4 + 14x^3 + 70x^2 - 100x + 80)\pi^2}{48x^2} \right. \\
 & - \left. \frac{(154x^3 - 421x^2 + 460x - 328)}{16x^2} \right) \ln^2(x) + \left[ -\frac{(27x^4 - 30x^3 - 16x^2 + 54x - 32)}{24x^2} \ln^3(1-x) \right. \\
 & - \left. \frac{(12x^4 - 7x^3 - 29x^2 + 42x - 24)}{8x^2} \ln^2(1-x) + \left( -\frac{(35x^4 + 2x^3 - 82x^2 + 86x - 48)\pi^2}{24x^2} \right. \right. \\
 & + \left. \left. \frac{(178x - 155x^2 + 44x^3 - 128)}{8x^2} \right) \ln(1-x) + \frac{(12x^4 - 100x^3 + 74x^2 - 11x + 24)\pi^2}{24x^2} \right. \\
 & + \left. \frac{(56x^4 + 165x^2 - 103x^3 - 121x + 68)}{6x^2} \zeta(3) - \frac{(599x^2 - 582x - 302x^3 + 448 + 128x^4)}{8x^2} \right] \ln(x) \\
 & + \frac{(3x^4 + 14x^3 - 18x^2 + 26x - 5)}{96x^2} \ln(1-x)^4 - \frac{(14x^2 + 4x^4 - 9x^3 - 9x + 4)}{8x^2} \ln^3(1-x) \\
 & + \left( -\frac{(5x^4 - 42x^3 + 92x^2 - 78x + 26)}{48x^2} \pi^2 - \frac{(-23x + 21 + 21x^2)}{8x} \right) \ln^2(1-x) \\
 & + \left( \frac{(23x^3 - 137x^2 + 134x - 60)}{24x^2} \pi^2 - \frac{(56x^4 + 165x^2 - 103x^3 - 121x + 68)}{6x^2} \zeta(3) \right. \\
 & + \left. \frac{(115x^2 - 76x - 76x^3 + 64 + 64x^4)}{4x^2} \right) \ln(1-x) + \frac{(-260 + 742x - 724x^2 + 34x^3 + 257x^4)}{1440x^2} \pi^4 \\
 & - \frac{(229x^2 - 200x^3 + 82x^4 - 98x + 82)}{48x^2} \pi^2 - \frac{(-295x - 241x^3 + 386x^2 + 168x^4 + 204)}{12x^2} \zeta(3) \\
 & + \frac{(-464x^3 + 288x^4 + 599x^2 + 288 - 464x)}{4x^2}. \tag{B.1}
 \end{aligned}$$

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